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GENERALIZED FROBENIUS GROUPS

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ABSTRACT

A pair (G, K) in which G is a finite group and $K \lhd G$, $1 < K < G$, is said to satisfy (F2) if $|C_G(x)| = |C_{G/K}(xK)|$ for all $x \in G \setminus K$. First we survey all the examples known to us of such pairs in which G is neither a p-group nor a Frobenius group with Frobenius kernel K. Then we show that under certain restrictions these are, essentially, all the possible examples.

1. Introduction

For our purpose, the most pertinent definition of Alan Camina's generalization of Frobenius groups is:

(F2) *the group G has a normal subgroup K, $1 < K < G$, such that if $x \in G \setminus K$ and $z \in K$, then $[x, y] = z$ for some $y \in G$.*

See [1]. All groups in this paper will be finite.

In [1] Camina presents two main results. One is a character-theoretic equivalent (F1) of (F2), which is not to be used in this paper. His other principal result states that if (G, K) *has* (F2) (that is G satisfies (F2) with respect to the normal subgroup K) then either K is a p-group or G/K is a p-group or else G is a Frobenius group with a Frobenius kernel K.

Knowledge of the p-groups with (F2) is meager. Most of it is contained in [4]. In particular, examples not of class 2 are few. All known examples are of class 2 or 3 or else have $K = Z(G)$, the center of G.

Nevertheless, one might hope for some description of the pairs (G, K) which have (F2) in terms of two seemingly basic components — the Frobenius groups, and the (F2)-pairs (G, K) in which G is a p-group. That is the problem to which we address ourselves in this paper.

What results should one seek to prove? In §2, we survey the examples known to us. In what we shall call F2-pairs (G, K) of type I, in which G/K is a p-group,

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we have examples (with $p = 2$) derived from those Frobenius groups having Frobenius complement isomorphic to the quaternion group of order 8. *A reasonable conjecture would be that (G, K) of type I, are just like this — that there are no others.*

Pairs of type II have K a p-group. Here the structure is more diversified. Clearly, the Sylow p-subgroup p of G must play an essential role. If (G, K) is an F2-pair of type II, it is not true that (G, P) must have (F2), though it may well be true that P is always normal in G. *It might be conjectured that (P, K) must have* (F2). It is easy to show (see Lemma (4.2)), that if P is normal in G then (P, K) does indeed have (F2).

We succeeded in proving the above conjectures only under the additional conditions that G is p-solvable and P has nilpotency class less than or equal to 2. We can prove the conjectures also in the case that K is cyclic. These partial answers to the conjectures are consequences of Theorem 4.1, Theorem 5.1 and Lemma 5.3. We feel, however, that these partial results together with our examples are useful in directing attention to the general problem.

Before stating one of the results we introduce some notation. Let p be a prime. We say that a pair (G, K) *has* F2(p) if (G, K) has (F2) with either G/K or K a p-group, but G is neither a p-group nor a Frobenius group with Frobenius kernel K. By Camina's theorem if (G, K) has (F2) and G is neither a p-group nor a Frobenius group with kernel K, then (G, K) has F2(q) for some prime q. We denote by Q_8 the quaternion group of order 8. The rest of our notation is standard.

The following theorem is a consequence of Theorem 4.1, Theorem 5.1 and Lemma 5.3.

THEOREM. *Let (G, K) have* F2(p) *for some prime p. Suppose that* either *K is cyclic*, or *G is p-solvable with a Sylow p-subgroup of nilpotency class at most* 2. *Then one of the following holds*:

(i) *$p = 2$, G is a Frobenius group with a Frobenius kernel N with $|K : N| = 2$ and a Frobenius complement isomorphic to Q_8.*

(ii) *If P is a Sylow p-subgroup of G, then $p \lhd G$ and (P, K) has* (F2). *Further, G is a semidirect product $G = HP$ and HK is a Frobenius group with Frobenius kernel K.*

For pair (G, K) that has F2(p) with K a p-group, some other conditions imply (ii), see section 4. If G/K is a p-group in the theorem with the Sylow p-subgroup of G having nilpotency class equal to 2, then it can be shown that (i) holds without assuming p-solvability (see Theorem 5.1). Also if (G, K) has F2 with

G/K a p-group, $p \neq 2$, and the nilpotency class of a Sylow p-subgroup of G is equal to 3, it can be proved that G has a normal p-complement. See Theorem 5.5.

2. Survey of examples

In this section we describe the examples known (to us) of pairs (G, K) that have F2(p). Camina states that he knows of only one example. It has $|G| = 72$ and $|G/K| = 4$; see the remarks following his theorem 2 in [1]. However, it is easy to see that any Frobenius group with Frobenius complement isomorphic to Q_8 gives similar examples with $|G/K| = 4$.

LEMMA 2.1. *If* (G, N) *and* $(G/N, K/N)$ *have* (F2) *then* (G, K) *has* (F2).

PROOF. Let $x \in G \setminus K$ and $z \in K$. By assumption there exists $y_1 \in G$ and $n \in N$ such that $[x, y_1] = zn$. Also, there exists $y_2 \in G$ such that $[x, y_2] = n^{-y_1^{-1}}$. Then:

$$[x, y_2 y_1] = [x, y_1][x, y_2]^{y_1} = zn \cdot (n^{-y_1^{-1}})^{y_1} = z,$$

as required.

EXAMPLE 1. Let G be a Frobenius group with Frobenius kernel N and $G/N \cong Q_8$. Let K be the subgroup of index 4 in G. Both (G, N) and $(G/N, K/N)$ have (F2). So by Lemma 2.1 (G, K) has (F2). The example is (G, K). There are of course many possibilities for N. We can have $N = \mathbf{Z}_p \times \mathbf{Z}_p$ where p is an odd prime with integers α, β satisfying $\alpha^2 + \beta^2 + 1 \equiv 0 \pmod{p}$ and generators of Q_8 acting on N according to the matrices $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \alpha & \beta \\ \beta & -\alpha \end{smallmatrix}\right)$. Camina's example of order 72 is the same with $p = 3$.

We know of no further examples of pairs (G, K) having F2(p) in which G/K is a p-group. The situation is different with regard to pairs in which K is a p-group. The following simple lemma is illuminating.

LEMMA 2.2. *Let* $G = PT$ *where* P, T *are subgroups of* G *with* $Z(P) = K \lhd G$ *and* $K \lneqq T$. *If* (P, K) *has* (F2) *and if* T *is a Frobenius group with Frobenius kernel* K *then* (G, K) *has* (F2).

PROOF. Take $x \in G \setminus K$ and $z \in K$. We have $x = x_1 x_2$ with $x_1 \in P$, $x_2 \in T$. If $x_2 \in K$, then $x \in P$ and $[x, y] = z$ for some $y \in P$ as (P, K) has (F2). So we may take $x_2 \in T \setminus K$. Since T is a Frobenius group with Frobenius kernel K, x_2 induces a regular automorphism of K. So $[x_2, k] = z$ for some $k \in K$. Then $[x, k] = z$ as $[x_1, k] = 1$.

EXAMPLE 2. Let $P = \langle a, b \rangle$ be a non-abelian of order 7^3 of exponent 7 and let $H_0 = \langle x \rangle$ of order 3. We construct the example $G = PH_0$ with $P \lhd G$ by having $a^x = a^2$, $b^x = b^2$. Note that $P' = Z(P) = \langle c \rangle$, $c = [a, b]$, $c^x = c^4$ ([2] p. 19) and that $\langle c, x \rangle = H$ is a Frobenius group with Frobenius kernel $K = \langle c \rangle$. Trivially (P, K) has (F2). So by Lemma 2.2 (G, K) has (F2). We even have G a Frobenius group with Frobenius kernel P.

Many variations on Example 2 are possible. The primes 3, 7, can be changed, P can be replaced by some more complicated group such as those described in [4], etc.

EXAMPLE 3. Let p be an odd prime and let β be a primitive $(p-1)p^n$-th root of unity modulo p^{n+1} $(n \geqq 1)$. Example 3 is $G = \langle a, b \mid a^{p^{n+1}} = b^{(p-1)p^n} = 1, a^b = a^\beta \rangle$, the holomorph of the cyclic group of order p^{n+1}. We put

$$P = \langle a, b^{p-1} \rangle, \quad K = \langle a^{p^n} \rangle, \quad H = \langle a^{p^n}, b^{p^n} \rangle.$$

The requirements of Lemma 2.2 are satisfied, for (P, K) is a stock example of a p-group with (F2) (see introduction of [4]) and H is a standard Frobenius group with kernel K. Also, $K = Z(P) \lhd G$. Therefore, (G, K) has F2(p). In this example we have that $P \lhd G$ without (G, P) having (F2) (unlike Example 2). Note that the Frobenius complement $\langle b^{p^n} \rangle$ of H even centralizes the nontrivial subgroup $\langle b^{p-1} \rangle$ of P.

EXAMPLE 4. We start with $P = \langle a_1, a_2, a_3, a_4 \rangle$, a group of class 2 and exponent 3 with $K = G' = Z(G) = \langle c, d \rangle$, $|G : K| = 3^4$, $|K| = 3^2$, having the commutator relations: $[a_1, a_2] = 1$, $[a_1, a_3] = c$, $[a_1, a_4] = d$, $[a_2, a_3] = d^{-1}$, $[a_2, a_4] = c$, $[a_3, a_4] = 1$. The criterion of theorem 3.1 of [4] easily establishes that (P, K) has (F2). Example 4 is to be $G = PQ$ where $Q = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2 \rangle \cong Q_8$. We shall have $P \lhd G$ once we have stated how Q acts on P. The action of α is given by: $a_1 \to a_2^{-1}$, $a_2 \to a_1$, $a_3 \to a_3^{-1}$, $a_4 \to a_4^{-1}$. It can be checked that automorphism α of P is defined thereby, and that α acts on K as follows: $c \to d^{-1}$, $d \to c$. The action of β is given by: $a_1 \to a_1 a_2^{-1}$, $a_2 \to a_1^{-1} a_2^{-1}$, $a_3 \to a_3^{-1}$, $a_4 \to a_4$, from which it follows that $c \to c^{-1} d^{-1}$, $d \to c^{-1} d$. It can be verified also that $\alpha^2 = \beta^2 = (\alpha\beta)^2$.

Then KQ is a Frobenius group with kernel K (KQ is in fact Camina's example of order 72). So by Lemma 2.2 (G, K) has (F2).

Example 4 incorporates most of the discouraging features of Example 3, and then some more. We have $P \lhd G$ but (G, P) does not have (F2). We have K noncyclic. We have the Frobenius complement Q of KQ acting on P with nontrivial fixed points. Note that G/P is noncyclic in this case. The only role that

Q_8 plays here is as a Frobenius complement; we do not use the fact that $(Q, Z(Q))$ has (F2). It may be claimed that Example 4 is reasonably complicated.

We know of no examples of pair (G, K) having F2(p) with K a p-group, that is not described by Lemma 2.2.

3. Some general preliminary results

PROPOSITION 3.1. *Let G be a group with normal subgroup K, $1 < K < G$. Then the following conditions on (G, K) are equivalent.*

(a) *(G, K) has* (F2).

(b) *If $g \in G \setminus K$ and $h \in K$ then g is conjugate in G to gh.*

(c) *If $x \in G \setminus K$, then $|C_G(x)| = |C_{G/K}(xK)|$.*

(d) *If aK and bK are conjugate in G/K and are nontrivial then a and b are conjugate in G.*

(e) *If X is an irreducible character of G with $K \nsubseteq \operatorname{Ker} X$, then $X(x) = 0$ for all $x \in G \setminus K$.*

PROOF. The equivalence of (a) and (b) is trivial and that of (c) and (e) follows the proof of corollary (2.24) of [3]. To show that (a) and (c) are equivalent let

$$C_G(a, K) = \langle x \in G \mid [a, x] \in K \rangle, \qquad \text{for all } a \in G \setminus K.$$

The following statements are equivalent for any $x_1, x_2 \in C_G(a, K)$: (i) $[a, x_1] = [a, x_2]$; (ii) $[a, x_1 x_2^{-1}] = 1$; (iii) $C_G(a)x_1 = C_G(a)x_2$. So, there is a one-to-one correspondence between $\{C_G(a)x \mid x \in C_G(a, K)\}$ and the subset of K $\{[a, x] \mid x \in C_G(a, K)\}$. Hence $|C_G(a, K) : C_G(a)| \leqq |K|$ and equality occurs if and only if for all $k \in K$ there exists $x \in G$ with $[a, x] = k$. Thus, equality occurs for all $a \in G \setminus K$ if and only if (G, K) has (F2). As $C_{G/K}(aK) \cong C_G(a, K)/K$, the equivalence of (a) and (c) is established. To see that (a) implies (d) see the proof of lemma 1 of [1]. Finally if (d) holds and $a \in G \setminus K$, $z \in K$ then $aK = azK$ implies that a is conjugate to az which is statement (b).

We note that other character-theoretic conditions equivalent to the above can be found in [1], explicitly in Theorem 1 and implicitly in its proof. Next we generalize Camina's necessary and sufficient condition for an (F2)-pair (G, K) to be G, a Frobenius group with Frobenius kernel K.

PROPOSITION 3.2. *Let (G, K) have* (F2). *Then G is a Frobenius group with Frobenius kernel K if and only if G splits over K.*

PROOF. To prove the non-trivial part we suppose that (G, K) has (F2) and that $G = HK$ with $H \cap K = 1$. Take $x \in H$ and $z \in K$. By (F2) there exists

$y \in G$ with $[x, y] = z$. Write $y = y_1 y_2$ where $y_1 \in H$, $y_2 \in K$. Then $z = [x, y] = [x, y_1 y_2] = [x, y_2][x, y_1]^{y_2}$. As $K \lhd G$ we get that $[x, y_2] \in K$ so that $[x, y_1] \in K$. But $x \in H$, $y_1 \in H$. So $[x, y_1] \in H \cap K = 1$ and consequently $[x, y_2] = z$, $y_2 \in K$. Thus

$$\{[x, g] \mid g \in K\} = K \qquad \text{for all } x \in G \setminus K.$$

So $[x, g] \neq 1$ for $x \in G \setminus K$ and $g \in K \setminus 1$. This implies that H acts fixed-point-freely on K and the proposition follows.

Corollary 3.3 (Camina [1], proposition 1). *Let (G, K) have* (F2). *Then G is a Frobenius group with Frobenius kernel K if and only if* $(|G/K|, |K|) = 1$.

Proof. Follows from Proposition 3.2 and the Schur–Zassenhaus Theorem.

Proposition 3.4. *Let (G, K) have* F2(p) *for some prime p and let P be any Sylow p-subgroup of G. Then $Z(P) \leqq K$.*

Proof. By Corollary 3.3, $|K|_p \neq 1$. Let $x \in Z(P)$ and assume that $x \notin K$. Then the number of conjugates of x is a multiple of $|K|$ (by Proposition 3.1, (b)) and so $|C_G(x)|$ divides $|G/K|$. It follows that $|P| = |C_G(x)|_p$ divides $|G|_p / |K|_p < |P|$, a contradiction.

Corollary 3.5. *If (G, K) has* F2(p) *for some prime p and P is a Sylow p-subgroup of G, then P is not abelian.*

Proof. Follows from the previous proposition and the fact that p divides $(|G/K|, |K|)$.

Lemma 3.6. *Let the pair (G, K) have* F2 *and let $G = HK$. Then $(H, H \cap K)$ has* F2.

Proof. Let $a \in H \setminus K$. Then $|C_H(a)| \leqq |C_G(a)| = |C_{G/K}(aK)| = |C_{H/H \cap K}(a(H \cap K))| \leqq |C_H(a)|$. Thus $|C_H(a)| = |C_{H/H \cap K}(a(H \cap K))|$ and the lemma follows.

4. F2(P)-pairs with K a p-group

In this section we assume that (G, K) has F2(p) for some prime p. All the examples known to us have the structure described by Lemma 2.2. Our main result in this section is a theorem giving necessary conditions for a Sylow p-subgroup of G to be normal in G and for G to have a structure like in Lemma 2.2.

THEOREM 4.1. *Let (G, K) have* F2(p) *with K a p-group, for some prime p. Let P be a Sylow p-subgroup of G. Assume that one of the following conditions holds:*

(i) *G is p-solvable and P/K is abelian.*

(ii) *G is p-solvable and the nilpotency class of P is at most* 2.

(iii) *K is cyclic.*

Then $P \lhd G$ and (P, K) have (F2). *Further, G is a semidirect product $G = HP$ and HK is a Frobenius group with Frobenius kernel K. Also in case* (ii) *we have $K = Z(P)$.*

The proof will be given following a series of lemmas.

LEMMA 4.2. *Let (G, K) have* F2(p) *with K a p-group, for some prime p. Let $P \in \mathrm{Syl}_p(G)$. If $P \lhd G$, then (P, K) has* (F2).

PROOF. Let $x \in P \setminus K$. By Proposition 3.1, (c), it suffices to show that $|C_{P/K}(xK)| = |C_P(x)|$. As $P \lhd G$ we get that $P \cap C_G(x) \in \mathrm{Syl}_p(C_G(x))$ and $(P/K) \cap C_{G/K}(xK) \in \mathrm{Syl}_p(C_{G/K}(xK))$. Now the (F2)-property implies that $|C_G(x)|_p = |C_{G/K}(xK)|_p$ and so $|C_G(x)|_p = |P \cap C_G(x)| = |C_P(x)| = |C_{G/K}(xK)|_p = |(P/K) \cap C_{G/K}(xK)| = |C_{P/K}(xK)|$, as needed.

LEMMA 4.3. *Let (G, K) have* F2(p) *with K a p-group, for some prime p. If T is any p'-subgroup of G then TK is a Frobenius group with Frobenius kernel K and complement T. In particular every Sylow subgroup of T is either cyclic or generalized quaternion.*

PROOF. Let $x \in T$ and $g \in K$, $g \neq 1$. By Proposition 3.1, (b), x and xg have the same order and so $[x, g] \neq 1$. Therefore, T acts fixed-point-freely on K and the lemma follows.

LEMMA 4.4. *Let (G, K) have* F2(p) *with K a p-group, for some prime p. Then $O_{p'}(G/K) = 1$.*

PROOF. Assume the contrary and let L/K be a minimal normal subgroup of G/K with $L/K \subseteq O_{p'}(G/K)$. As $(|L/K|, |K|) = 1$, L splits over K so that $L = VK$ for some subgroup V. Also $V \cap K = 1$. Hence $V \cong L/K$ is a p'-subgroup of G. By Lemma 4.3 all the Sylow subgroups of G are either cyclic or generalized quaternion and hence V is solvable (by the Suzuki–Brauer theorem). So $V \cong L/K$ is a q-group for some prime $q \neq p$. In fact $V \in \mathrm{Syl}_q(L)$. As $L \lhd G$, the Frattini argument yields that $G = N_G(V)L = N_G(V)K$.

By Lemma 4.3, VK is a Frobenius group with Frobenius kernel K and complement V. In particular $V \neq V^g$ for all $g \in K$ and so $K \cap N_G(V) = 1$. We conclude that $G = N_G(V)K$ splits over K. Now Proposition 3.2 implies that (G, K) does not have F2(P) (although it has (F2)), a contradiction.

LEMMA 4.5. *Let (G, K) have* F2(p) *with K a p-group, for some prime p. Let $P \in \mathrm{Syl}_p(G)$. If $P^x \neq P^y$ for some x, $y \in G$, then $Z(P^x) \cap Z(P^y) = 1$.*

PROOF. Assume the contrary and let $1 \neq z \in Z(P^x) \cap Z(P^y)$. Then $z \in Z(\langle P^x, P^y \rangle) \cap K$, by Proposition 3.4. As $P^x \neq P^y$, $\langle P^x, P^y \rangle$ is not a p-group. Let T be any p'-subgroup of $\langle P^x, P^y \rangle$. Then T centralizes $z \in K$, contradicting Lemma 4.3.

PROOF OF THEOREM 4.1. It suffices to show that $P \lhd G$. For if $P \lhd G$, G splits over P so that $G = HP$, $H \cap P = 1$ and H is a p'-subgroup. Then the theorem follows from Lemma 4.2, Lemma 4.3 and lemma 2.1 of [4].

Assume that G is p-solvable, then G/K is p-solvable and as $O_{p'}(G/K) = 1$ (by Lemma 4.4) we get that $C_{G/K}(O_p(G/K)) \leqq O_p(G/K)$ (by the Hall–Higman 1.2.3 Lemma).

To prove that $P \lhd G$ in case (i) we note that if P/K is abelian then: $O_p(G/K) \leqq P/K \leqq C_{G/K}(O_p(G/K)) \leqq O_p(G/K)$. Hence, $O_p(G/K) = P/K \lhd G/K$ forcing $P \lhd G$.

If (ii) holds, then P/K is abelian as $Z(P) \subseteq K$ (by Proposition 3.4). Thus, $P \lhd G$ by (i).

Finally assume that (iii) holds, that is, K is cyclic. Let $x \in G$. If $P^x \neq P$, then by Lemma 4.5 $Z(P) \cap Z(P^x) = 1$ and by Proposition 3.4 $\langle Z(P), Z(P^x) \rangle \subseteq K$, contradicting the fact that K is a cyclic p-group. Hence $P = P^x$ and $P \lhd G$.

5. F2(p)-groups with G/K a p-group

In this section we prove the following theorem which, together with Theorem 4.1 and Lemma 5.3, yields a proof to the Theorem stated in the introduction.

THEOREM 5.1. *Let (G, K) have* F2(p) *with G/K a p-group, for some prime p. If a Sylow p-subgroup of G has class at most* 2, *then G is a Frobenius group, the Frobenius kernel has index* 2 *in K and the Frobenius complement is isomorphic to Q_8. In particular $p = 2$ and $|G/K| = 4$.*

In our first lemma we prove part of Theorem 5.1.

LEMMA 5.2. *Suppose that (G, K) satisfies the assumptions of Theorem* 5.1. *Then* (a) *G has a normal p-complement and* (b) *$p = 2$ and $|G/K| = 4$.*

PROOF. Let $P \in \mathrm{Syl}_p(G)$ and $Q = P \cap K$. By Corollary 3.5, the nilpotency class of P is equal to 2. As $G = KP$, Lemma 3.6 implies that (P, Q) has F2 and using [4] we get that $Q = Z(P)$. Also,

(0) $P^g \cap K = Z(P^g)$ for all $g \in G$.

(a) By the Frattini argument $G = HK$, where $H = N_G(Q)$. Applying Lemma 3.6 again yields that $(H, H \cap K)$ has F2. Let $C = C_G(Q)$. Then $P \leqq C \lhd H$. If $a \in H \cap K$ and $x \in P - H \cap K$, the F2-property of $(H, H \cap K)$ implies that $a = [x, h]$ for some $h \in H$. Therefore $H \cap K \leqq [P, H] \leqq C$ and consequently $[H \cap K, Q] = 1$. Clearly $H = (H \cap K)P$ and hence $Q \leqq Z(H)$. Now, $Q \in \mathrm{Syl}_p(K)$ and the Burnside theorem ([2] p. 252) implies that K has a normal p-complement which is clearly a normal p-complement of G.

(b) Let G be a counterexample of minimal order to statement (b) and let R be the normal p-complement of G whose existence is proved in part (a). By Corollary 3.3, $|K|_p \neq 1$ and so $R < K$. Let N be a minimal normal subgroup of G contained in R. Then $N < K$ and so $(G/N, K/N)$ has F2. If $(G/N, K/N)$ has F2(p) we have, by induction, that $p = 2$ and $|G/K| = |G/N : K/N| = 4$, a contradiction. Therefore, either G/N is a Frobenius group with Frobenius kernel K/N or G/N is a p-group. In the former case G/K is isomorphic to a Frobenius complement and hence G/K is either cyclic or generalized quaternion. This contradicts lemma 7 of [1] since $G/K = PK/K \simeq P/P \cap K$. We conclude that G/N is a p-group.

It follows that N is a p'-group and we have the following semidirect products: $G = NP$, $K = N(P \cap K)$. We quote from [4] (corollary 2.4 with its proof and section 3) some facts about G/N. First $K/N = Z(G/N)$ is elementary abelian. Also $G/K \simeq (G/N)/(K/N)$ is elementary abelian. Further $|K/N| = p^n$, $|G/K| = p^{2m}$ with $m \geqq n \geqq 1$.

Let $(G/N)/(K/N) = \langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle \times \cdots \times \langle \bar{a}_{2m} \rangle$, where $\bar{a}_i = a_i K/N$. Then $a_1^p, a_2^p, \cdots, a_{2m}^p$ are linearly dependent elements of K/N. We therefore have $a_1^{p\alpha_1} a_2^{p\alpha_2} \cdots a_{2m}^{p\alpha_{2m}} = 1$ for some $\alpha_1, \alpha_2, \cdots, \alpha_{2m}$ with $0 \leqq \alpha_i < p$ and not every $\alpha_i = 0$. If $p \neq 2$, the element $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_{2m}^{\alpha_{2m}}$ has order p (see [2], p. 183) and lies in $(G/N) \setminus (K/N)$.

Therefore, if $p \neq 2$ we can find an element $xN \in (G/N) \setminus (K/N)$, xN of order p. Hence the p-part of x, x_p, lies in $G \setminus K$ and has order p. By the (F2) property x acts fixed-point-freely on N and by [2] (p. 337), N is nilpotent. If $p = 2$, N is solvable. As N is minimal normal we get that in all cases N is an elementary abelian q-group with $q \neq p$. The proof now breaks into several steps.

(1) $N_G(P) = P$. *Proof.* $N_G(P) = N_N(P) \times P$ and the (F2) property implies that $N_N(P) = 1$ because no element of $P \setminus K$ commutes with a p'-element.

(2) $Z(G)=1$. *Proof.* Assume $Z=Z(G)\neq 1$. By the (F2) property $Z(G)$ is a p-group so $Z\leqq Z(P)<K$ (see Proposition 3.4). Then $(G/Z, K/Z)$ has (F2) and G/Z is not a p-group. If $(G/Z, K/Z)$ has F2(p) we get that $|G/K|=4$, by induction, a contradiction. Hence G/Z is a Frobenius group with G/K isomorphic to its Frobenius complement. As before, this contradicts lemma 7 of [1].

(3) If $P\neq P^x$ then $\langle P, P^x\rangle = G$. *Proof.* Set $U=\langle P, P^x\rangle$. Then $U=(U\cap N)P$ with $U\cap N\lhd U$. As N is abelian, $U\cap N\lhd N$ so that $G=NP=NU\leqq N_G(U\cap N)$. Hence $U\cap N\lhd G$. As N is minimal normal and $U\neq P$, we get that $U\cap N=N$ forcing $U=(U\cap N)P=NP=G$.

(4) If $P\neq P^g$ then $P\cap P^g\leqq K$. *Proof.* Suppose not and let $x\in P\cap P^g\setminus K$. Then $\langle Z(P), Z(P^g)\rangle\leqq C_G(x)$ and by (0) we have $L=\langle P\cap K, P^g\cap K\rangle\leqq C_G(x)$. By the (F2) property L is a p-group and as $P\cap K\in \mathrm{Syl}_p(K)$ we get that $P\cap K=P^g\cap K$. Again (0) implies that $Z(P)=Z(P^g)$ so that $Z(P)\subseteq C_G(\langle P, P^g\rangle)=Z(G)$, by (3). This contradicts (2).

(5) If $P\neq P^g$, then $P\cap P^g=1$. *Proof.* By (0), (2), (3) and (4) we get $P\cap P^g\subseteq K$, so that $P\cap P^g=(P\cap K)\cap(P^g\cap K)=Z(P)\cap Z(P^g)\leqq C_G(\langle P, P^g\rangle)=Z(G)=1$.

Steps (1) and (5) imply that G is a Frobenius group with Frobenius complement P and Frobenius kernel N. By lemma 7 of [1], $P/P\cap K$ is not cyclic and hence P is a generalized quaternion group of nilpotency class equal to 2. Therefore $P\cong Q_8$. Recall that K/N is an elementary abelian subgroup of $G/N\cong Q_8$. Thus $|K/N|=2$ and $|G/K|=4$, a final contradiction.

LEMMA 5.3. *Let (G,K) have* (F2) *with G/K a p-group. Assume that G contains a normal subgroup $1\neq N<K$ with K/N a p'-group. Then G is a Frobenius group with Frobenius kernel K.*

PROOF. As $(G/N, K/N)$ has (F2), Corollary 3.3 implies that G/N is a Frobenius group with G/K isomorphic to its Frobenius complement. If $(|G/K|,|K|)\neq 1$, we get a contradiction to lemma 7 of [1] as in the proof of the previous lemma. Hence $(|G/K|,|K|)=1$ and Lemma 5.3 follows from Corollary 3.3.

LEMMA 5.4. *Let (G,K) satisfy the assumptions of Theorem* 5.1. *Assume that N is a normal p'-Hall subgroup of G. Then $p=2$ and $G/N\cong Q_8$.*

PROOF. Clearly $1<N<K$ and $G/N\cong P\in \mathrm{Syl}_p(G)$. By Lemma 5.2, $p=2$ and $|G/K|=4$. As $(G/N, K/N)$ has (F2) with G/N nilpotent of class 2 we get

from [4] (corollary 2.4 and its proof and theorems 3.1, 3.2) that K/N and $(G/N)/(K/N)$ are elementary abelian with $|K/N| = 2^n$, $|G/K| = 2^m$ with $2 = m \geqq 2n$. Hence $n = 1$, $|K/N| = 2$ and $|P| = |G/N| = 8$. By Corollary 3.5, P is not abelian. If P is a dihedral group of order 8, then an involution $t \in P \setminus P \cap K$ can be found as $G = NP$ and $|P \cap K| = 2$. By lemma 4 of [1], K is nilpotent so that Lemma 5.3 implies that G is a Frobenius group with Frobenius kernel K. This contradiction yields that $P \cong Q_8$.

Proof of Theorem 5.1. By Lemma 5.2, $G = NT$ where $T \in \mathrm{Syl}_2(G)$ and N is a $2'$-group, and by Lemma 5.4, $T \simeq Q_8$. Clearly $|K : N| = 2$. We still have to show that T acts fixed-point-freely on N. Set $T = \langle a, b \mid a^2 = b^2 = (ab)^2 = t \rangle$ and $F = C_N(t)$. If $x \in F$ then $[x^a, t] = [x, t]^a = 1$ so a induces an automorphism α on F by conjugation. As $a \in T \setminus K$, α is fixed-point-free on F (by the F2-property) and since $a^2 = t$, $\alpha^2 = 1$. By [2], p. 336 $\alpha(x) = x^{-1}$ for all $x \in F$. Similarly we get $x^a = x^b = x^{ab} = x^{-1}$ for all $x \in F$. This is impossible unless $F = 1$. Therefore t (as well as all elements of $T \setminus K$) acts fixed-point-freely on N.

The next result (and its proof) is due to the referee.

Theorem 5.5. *Let (G, K) have* F2(p) *with G/K a p-group, and let $P \in \mathrm{Syl}_p(G)$ have class* 3. *If $p \neq 2$ then G has a normal p-complement.*

Proof. As $G = KP$, Lemma 3.6 implies that $(P, P \cap K)$ has F2. By [4] (lemma 2.1) either $P \cap K = Z(P)$ or $P \cap K = Z_2(P) = [P, P]$. As $\mathrm{cl}(P) = 3$, $P \cap K$ is abelian in both cases. Now the theorem follows from the next theorem.

Theorem 5.6. *Let (G, K) have* F2(p) *with G/K a p-group, $p \neq 2$, and let $P \in \mathrm{Syl}_p(G)$. If $P \cap K$ is abelian then G has a normal p-complement.*

Proof. Let G be a counterexample of minimal order. Set $Q = P \cap K$, $C = C_G(Z(P))$ and $H = N_G(J(P))$ where $J(P)$ is the Thompson subgroup (see [5], p. 289 for definition of $J(P)$). Clearly $P \subseteq C$ and $P \subseteq H$ so that $G = KC = KH$. Lemma 3.6 implies that $(C, C \cap K)$ and $(H, H \cap K)$ have F2. By Proposition 3.2 we have that $Z(P) \subseteq C \cap K$ and so $(C/Z(P), (C \cap K)/Z(P))$ has F2. By induction $C/Z(P)$ has a normal p-complement $R/Z(P)$ (note that if this pair doesn't have F2(p), then clearly $R/Z(P)$ exists). As $R \subseteq C$, we have that $Z(P) \subseteq Z(R)$ and so the Schur–Zassenhaus theorem implies that $R = U \times Z(P)$, where U is clearly a normal p-complement of C.

Let's go back to H. If $H < G$, then induction implies that H has a normal p-complement and then by Thompson's theorem ([5], p. 289) so has G, a contradiction. Hence $H = G$ so that $J(P) \lhd G$. As $J(P) \subseteq C$, we get that

$[U, J(P)] = 1$. Now the F2-property implies that $J(P) \subseteq K$ and hence $J(P) \subseteq Q$. But Q is abelian so that $J(P) = Q$ and in particular $Q \lhd G$. It follows that $(G/Q, K/Q)$ has F2 and since $Q \in \mathrm{Syl}_p(K)$, Corollary 3.3 implies that G/Q is a Frobenius group with a complement isomorphic to G/K. Thus G/K is cyclic, contradicting lemma 7 of [1].

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